Computational complexity of iterated maps on points and sets

Christoph Spandl

Fakultät für Informatik
Universität der Bundeswehr München

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Discrete dynamical systems - point dynamics

- Let $M \subseteq \mathbb{R}^n$ be a *cuboid*, that is of the form $M = [a_, b_1] \times \ldots \times [a_n, b_n]$ with $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$.
- Let $f\colon M \to M$ be a *self mapping*, in particular a C^2-diffeomorphism.
- Then the pair (M, f) is called a *discrete dynamical system*.
- The dynamics is governed by the ***iteration equation***

$$x^{(k+1)} = f(x^{(k)})$$
$$x^{(0)} \in M.$$

- The second condition specifies the dynamics as an *initial value problem*.
- The initial value produces an *orbit* $(x^{(k)})_{k \in \mathbb{N}}$ under the dynamics.

Discrete dynamical systems - set dynamics

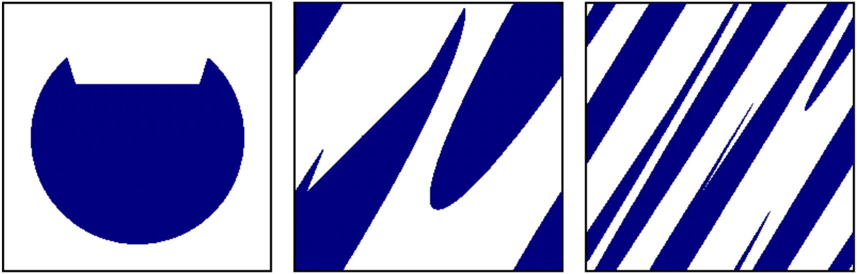
point dynamics - set dynamics

The standard definition as above describes a dynamics of ***points*** in $\mathbb{R}^n$. However, dynamics can also be formulated for ***sets*** of $\mathbb{R}^n$.

- Denote by C_M the set of all compact subsets $A \subseteq M$ of M.
- Generalize $f\colon M \to M$ to $f_C\colon C_M \to C_M$ generically by setting $f_C(A) := f[A]$ for all $A \in C_M$.
- Then the pair (C_M, f_C) defines a discrete dynamical system on sets.
- Note that if $f\colon M \to M$ is computable, then also $f_C\colon C_M \to C_M$ is computable.

Set dynamics - problems

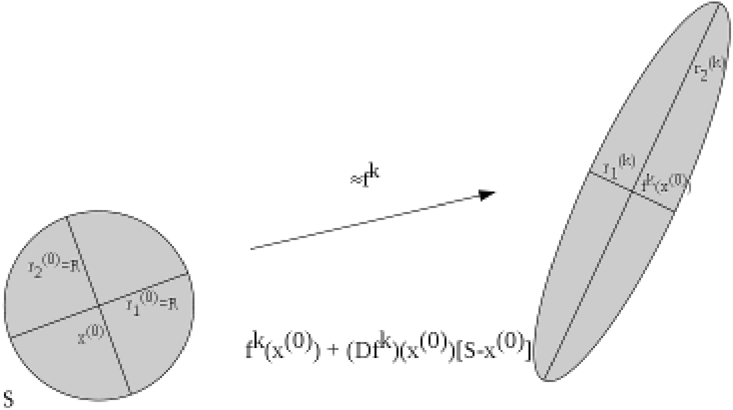
Set dynamics in the case of **mixing**: Arnold's cat map



- The map is ***area preserving***.
- The initial set is uniformly spread over the whole domain in a few iterations.
- The number of spheres covering $A^{(k)}$ for given accuracy is typically growing ***exponentially*** in the number of iterations.

Local set dynamics by linearization: cocycles (1)

Consider a sphere $S = B(x^{(0)}, r)$ and examine $(Df^k)(x^{(0)})$. For given k and r ***sufficiently small***, $f^k(x^{(0)}) + (Df^k)(x^{(0)})B(0, r) \approx f^k[S]$.



Local set dynamics by linearization: cocycles (2)

Lyapunov exponents

- The length of the axes of the ellipsoid $f^k(x^{(0)}) + (Df^k)(x^{(0)})B(0, r)$ are denoted by $r_1^{(k)}, \ldots, r_n^{(k)}$.
- The value $r_i^{(N)}$ measures the ***contraction*** or ***expansion*** near the orbit segment $(x^{(k)})_{k \leq N}$ along the ith principal axis.
- Then the ith *Lyapunov exponent* λ_i is given by

$$\lambda_i = \lim_{k \to \infty} \frac{\ln(r_i^{(k)})}{k}$$

if the limit exists.

Local set dynamics by linearization: cocycles (3)

This linearized map can be expressed in the framework of a dynamical system by a *cocycle*:

$$x^{(k+1)} = f(x^{(k)})$$
$$z^{(k+1)} = (Df)(x^{(k)}) \cdot z^{(k)}$$
$$x^{(0)} \in M, \quad z^{(0)} = \mathbb{1}$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. Note that $z^{(k)} = (Df^k)(x^{(0)})$.

From approximations to enclosures

- The above linearization ***approximates*** $f^k[S]$.
- It is only ***asymptotically exact*** for $R \to 0$, but not for $R > 0$.
- However, this approach can be made ***rigorous*** even for $R > 0$.
- In verified computing, *enclosures* are used.
- Here, this is a function $\overline{f}_C \colon C_M \to C_M$ satisfying

  $$f_C(A) \subseteq \overline{f}_C(A)$$

  for all $A \in C_M$.
- In the following it is convenient to restrict the domain of the enclosure to ***cuboids***.
- Let Q_M be the set of all cuboids $I \subseteq M$, then find an appropriate function $\overline{f}_Q \colon Q_M \to Q_M$ satisfying $f_C(A) \subseteq \overline{f}_Q(A)$ for all $A \in Q_M$.

Finding an appropriate enclosure (1)

- Use a Taylor polynomial with remainder term:

$$f_i(y) = f_i(x) + \sum_j \frac{\partial f_i}{\partial x_j}\big(x + \Theta_i(y - x)\big)(y_j - x_j)$$

for $x, y \in M$ with $\Theta_i \in [0,1]$, $i = 1, \ldots, n$.

- Furthermore assume a *Lipschitz condition*:

$$\left|\frac{\partial f_i}{\partial x_j}(y) - \frac{\partial f_i}{\partial x_j}(x)\right| \leq (L(I))_{ij} \cdot \|y - x\|_\infty$$

- Then

$$f_C(I) \subseteq f(x) + \big((Df)(x) + [-1,1] \cdot |I| \cdot L(I)\big)(I - x)$$

for all $I \in Q_M$ where $|I| = \sup_i(|I_i|) = \sup_i(b_i - a_i)$.

Finding an appropriate enclosure (2)

- Assume the following *normal form*

$$I = x + [-1, 1] \cdot e$$

for cuboids $I \in Q_M$, where $x \in M$, $e \in \mathbb{R}^n_+$.

- Then

$$f_C(I) \subseteq f(x) + [-1, 1] \cdot V(x, e) \cdot e$$

where

$$V(x, e) = |(Df)(x)| + 2\|e\|_\infty L(x + [-1, 1] \cdot e).$$

- Reformulation: let

$$CQ_M = \{(x, e) \in M \times \mathbb{R}^n_+ \mid x + [-1, 1] \cdot e \in Q_M\},$$

then define $\bar{f}_Q \colon CQ_M \to M \times \mathbb{R}^n_+$ by

$$\bar{f}_Q(x, e) = (f(x), V(x, e) \cdot e)\,.$$

The modified cocycle

This set dynamics can be formulated by a *modified cocycle*:

$$
\begin{aligned}
x^{(k+1)} &= f(x^{(k)}) \\
z^{(k+1)} &= V(x^{(k)}, e^{(k)}) \cdot z^{(k)} \\
x^{(0)} &\in M, \quad z^{(0)} = \mathbb{1}
\end{aligned}
$$

where

$$
\begin{aligned}
V(x,e) &= |(Df)(x)| + 2\|e\|_\infty L(x,e) \\
e^{(k)} &= z^{(k)} \cdot e^{(0)} \\
e^{(0)} &\in \mathbb{R}^n \; s.t. \; (x^{(0)}, e^{(0)}) \in CQ_M.
\end{aligned}
$$

The model of computation - representing reals

- We start with a *fixed point number system*:

$$\widehat{\mathbb{R}}(p,\beta) = \{x \in \mathbb{R} \mid \exists\, r, s \in \mathbb{Z} \,.\, x = s + r \cdot \beta^{-p} \ \wedge\ |r| \leq \beta^{p} - 1\}$$

- where $\beta \geq$ is the ***base*** and $p \geq 1$ the ***precision***.
- Then we allow *arbitrary precision*:

$$\widehat{\mathbb{R}}(\beta) = \bigcup_{p \geq 1} \widehat{\mathbb{R}}(p,\beta)$$

- A fixed point number $\hat{x} \in \widehat{\mathbb{R}}(p,\beta)$ *approximates* a real $x \in \mathbb{R}$, if

$$x \in \hat{x} + \beta^{-p}[-1,1].$$

- Any $x \in \mathbb{R}$ can be *represented* by a sequence $(\hat{x}_n)_{n \in \mathbb{N}}$ with $\hat{x}_n \in \widehat{\mathbb{R}}(p_n,\beta)$, each $\hat{x}_n$ approximating x and $\lim_{n \to \infty} p_n = \infty$.

The model of computation - representing functions

- Let $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ be given. A function $\hat{f} :\subseteq \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}$ is called an *approximation function* for f if:

$$\hat{x} \in \widehat{\mathbb{R}}^n \text{ approximates } x \in \mathrm{dom}(f) \implies \hat{f}(\hat{x}) \text{ approximates } f(x).$$

- We call $\hat{f}$ *approximation continuous* if for any $(\hat{x}_n)_{n\in\mathbb{N}}$:

$$(\hat{x}_n)_{n\in\mathbb{N}} \text{ representing } x \in \mathrm{dom}(f) \implies (\hat{f}(\hat{x}_n))_{n\in\mathbb{N}} \text{ representing } f(x)$$

- Since $\widehat{\mathbb{R}}$ is countable, define computability for $\hat{f} :\subseteq \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}$ by ***classical computability theory***.
- Then f is called *computable*, if it has a computable approximation continuous approximation function.

Further specifications and generalizations

- Generalization of an approximation of $x \in \mathbb{R}$:

$$(\hat{x}, \overline{e}) \in \widehat{\mathbb{R}}(p_1, \beta) \times \widehat{\mathbb{R}}(p_0, \beta) \text{ approx. } x \in \mathbb{R} \iff x \in \hat{x} + [-1, 1] \cdot \overline{e}$$

- ***Normal form*** for approximation functions $\hat{f} :\subseteq \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}$:

$$\hat{x} \in \widehat{\mathbb{R}}(p_1, \beta) \times \ldots \times \widehat{\mathbb{R}}(p_n, \beta) \implies \hat{f}(\hat{x}) \in \widehat{\mathbb{R}}(p_0, \beta)$$
$$p_0 = \min(p_1, \ldots, p_n)$$

- Normal form for approximations of ***self mappings*** $f : M \to M$:

$$\hat{x} \in \widehat{\mathbb{R}}(p_1, \beta) \times \ldots \times \widehat{\mathbb{R}}(p_n, \beta) \implies \hat{f}(\hat{x}) \in \widehat{\mathbb{R}}(p'_1, \beta) \times \ldots \times \widehat{\mathbb{R}}(p'_n, \beta)$$
$$\beta^{-p'} \leq \left| Q \cdot \beta^{-p} \right|$$

where $Q \in \mathbb{R}^{n \times n}$ is an ***orthogonal matrix***.

Point dynamics - formulating the algorithm (1)

- A ***finite segment*** $(x^{(k)})_{0\leq k\leq N}$ of lenght N of the *true orbit* is computed: a *pseudo orbit* $(\hat{x}^{(k)})_{0\leq k\leq N}$
- with demanded precision $p_1^o, \ldots, p_n^o$: $x_i^{(k)} \in \hat{x}_i^{(k)} + \beta^{-p_i^o}[-1, 1]$ for ***all*** $k \leq N$.
- In the above formulation:

$$x^{(k)} \in \hat{x}^{(k)} + [-1, 1] \cdot \overline{e}^{(k)}, \quad \overline{e}^{(k)} \leq \beta^{-p^o}.$$

- Since N is fixed, there ex. $p_1^s, \ldots, p_n^s \geq 1$ s.t. the above condition is fulfilled starting with

$$\hat{x}^{(0)} \in \widehat{\mathbb{R}}(p_1^s, \beta) \times \ldots \times \widehat{\mathbb{R}}(p_n^s, \beta)$$

and approximating f by

$$\hat{x}^{(k)} \in \widehat{\mathbb{R}}(p^{(k)}, \beta) \implies \hat{f}(\hat{x}) \in \widehat{\mathbb{R}}(p^{(k+1)}, \beta)$$
$$\beta^{-p^{(k+1)}} \leq \overline{e}^{(k+1)} \leq \left|Q^{(k+1)}\beta^{-p^s}\right|.$$

Point dynamics - formulating the algorithm (2)

- Using the above precision control for $\hat{f}$ and the ***modified cocycle*** for calculating the *error propagation* leads to

$$\overline{e}^{(k+1)} \geq V(\hat{x}^{(k)}, \overline{e}^{(k)})\overline{e}^{(k)} + |Q^{(k+1)}\beta^{-p^s}|$$

for the recursion of the error.

- Finally, by approximating Df by $\widehat{Df}$,

$$\overline{e}^{(k+1)} = \overline{V}(\hat{x}^{(k)}, \overline{e}^{(k)})\overline{e}^{(k)} + |Q^{(k+1)}|\beta^{-p^s}$$

is obtained where

$$\overline{V}(\hat{x}, \overline{e}) = |(\widehat{Df})(\hat{x})| + \|\overline{e}\|_\infty(2 \cdot \overline{L}(\hat{x}, \overline{e}) + E).$$

Set dynamics - differences in the algorithm

- By pairing $(x^{(k)}, z^{(k)})$, the modified cocycle can be viewed as a ***new dynamical system***: then

  set dynamics is reduced to point dynamics with ***different phase space***.
- ***Alternatively***, the pair $(\hat{x}^{(k)}, \overline{e}^{(k)})$ is interpreted as an *enclosure for cuboids*:

$$I^{(k)} = x^{(k)} + [-1, 1] \cdot e^{(k)} \subseteq \hat{x}^{(k)} + [-1, 1] \cdot \overline{e}^{(k)}.$$

- Then the error control has to be reinterpreted.
- But the resulting formulas are ***nearly the same*** as in the case of points.
- Only the ***interpretation*** is different.

Point dynamics - set dynamics (enclosures)

$$
\begin{aligned}
\hat{x}^{(k+1)} &= \hat{f}(\hat{x}^{(k)}) \\
\overline{z}^{(k+1)} &= \overline{V}(\hat{x}^{(k)}, \overline{e}^{(k)})\overline{z}^{(k)} + |Q^{(k+1)}| \\
\hat{x}^{(0)} &\in \widehat{\mathbb{R}}(p_1^s, \beta) \times \ldots \times \widehat{\mathbb{R}}(p_n^s, \beta) \\
\overline{V}(\hat{x}, \overline{e}) &= |(\widehat{Df})(\hat{x})| + \|\overline{e}\|_\infty (2 \cdot \overline{L}(\hat{x}, \overline{e}) + E) \\
\overline{e}^{(k)} &= \overline{z}^{(k)} \overline{e}^{(0)} \\
\overline{e}^{(0)} &= \alpha \cdot \beta^{-p^s}, \quad \overline{z}^{(0)} = \mathbb{1}
\end{aligned}
$$

- Even the algorithm can be expressed via a *modified cocycle*.
- $\alpha = 1$ in the case of points, $\alpha = 2$ in the case of cuboids.
- p^s is the initial precision for $\hat{x}^{(0)}$ in the case of points.
- In the case of cuboids, p^s determines an upper bound on the extent of $I^{(0)}$.

Computational complexity: loss of significance rates

Definition

Let $p_i^{min}(x, N, p^o)$ be the minimal precisions for p_i^s such that the demanded precision p^o for the pseudo orbit of length N is achieved when $x \in M$ is the initial condition.

- The *growth rate* of $p_i^{min}(x, N, p^o)$ is

$$\sigma(x, p^o) = \limsup_{N \to \infty} \frac{p^{min}(x, N, p^o)}{N}.$$

The *loss of significance rates* $\sigma : M \to \mathbb{R}^n$ are defined by

$$\sigma(x) = \lim_{p \to \infty} \sigma(x, p). \tag{1}$$

Loss of significance rate - main results (1)

An easy provable observation: the loss of significance rates are ***bounded***. A bit more effort: the loss of significance rates are bounded from below by the ***Lyapunov exponents.***

Proposition

Let (M, f) be a dynamical system, $x \in M$ and $\sigma(x)$ the loss of significance rates. Then there exist some $c \in \mathbb{R}^n_+$ such that $(0, \ldots, 0)^t \leq \sigma(x, p) \leq \sigma(x) \leq c$ holds for all precisions $p_1, \ldots, p_n \geq 1$.

Theorem

Let the notation as above. Then

$$\sigma_i(x) \geq \frac{1}{\ln(2)} \lambda_i(x)$$

holds for $i = 1, \ldots, n$ where $\lambda_i(x)$ is the ith Lyapunov exponent, if it exists.

Loss of significance rate - main results (2)

- The proof of the theorem is based on a QR-decomposition of the form

$$Q^{(k+1)}R^{(k+1)} = (Df)(x^{(k)})Q^{(k)}$$
$$Q^{(0)} = \mathbb{1}.$$

- The link between this QR-decomposition and the Lyapunov exponent is well established in the literature.
- Also $R^{(k)}$ and $Q^{(k)}$ are approximated by $\hat{R}^{(k)}$, $\hat{Q}^{(k)}$.
- Since in $\overline{e}^{(k)}$ not $(\widehat{Df})(.)$ but $|(\widehat{Df})(.)|$ is relevant the matrix multiplication of matrices of the form $|A|$ need to be done more elaborate to reduce ***overestimation of the error***.

Loss of significance rate - main results (3)

If some additional aspects concerning the *QR*-decomposition and the Lyapunov exponents turn out to be true (which actually have not been checked yet), the we also have:

Theorem

Let the notation as above. Then

$$\sigma_i(x) \leq \frac{1}{\ln(2)} \lambda_i(x)$$

holds for $i = 1, \ldots, n$.

Thus the Lyapunov exponents turn out also to be an ***upper bound*** on the loss of significance rates.